On rings with divided nil ideal: a survey

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Abstract. Let R be a commutative ring with $1 \neq 0$ and $\mathrm{Nil}(R)$ be its set of nilpotent elements. Recall that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In many articles, the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$ (Observe that if R is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then R is called a ϕ-ring. Recently, David Anderson and the author generalized the concept of Prüfer domains, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. In this paper, we state many of the main results on ϕ-rings.

Keywords. Prüfer ring, ϕ-Prüfer ring, Dedekind ring, ϕ-Dedekind ring, Krull ring, ϕ-Krull ring, Mori ring, ϕ-Mori ring, divided ring.

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1 Introduction

Note 1 Nil, Rad, Max, dim upright

Note 2 replace Bezout by Bézout?

Let R be a commutative ring with $1 \neq 0$ and $\mathrm{Nil}(R)$ be its set of nilpotent elements. Recall from [26] and [7] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [6], [8], [9], [10], and [11], the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. (Observe that if R is an integral domain, then $R \in \mathcal{H}$.) If $R \in \mathcal{H}$, then R is called a ϕ-ring. Recently, David Anderson and the author, [3] and [4], generalized the concept of Prüfer, Bezout domains, Dedekind domains, and Krull domains to the context of rings that are in the class $\mathcal{H}$. Also, Lucas and the author, [17], generalized the concept of Mori domain to the context of rings that are in the class $\mathcal{H}$. Yet, another paper by Dobbs and the author [14] investigated going-down ϕ-rings. In this paper, we state many of the main results on ϕ-rings.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, and $Z(R)$ denotes the set of zerodivisors of R. We start by recalling some background material. A non-zerodivisor of a ring R is called a *regular element* and an ideal of R is said to be *regular* if it contains a regular element. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then $\mathrm{Nil}(R) \subset I$. In particular, this holds if I is a regular ideal of a ring $R \in \mathcal{H}$.

Recall from [6] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $\mathrm{Ker}(\phi) \subseteq \mathrm{Nil}(R)$, $\mathrm{Nil}(T(R)) = \mathrm{Nil}(R)$, $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{\mathrm{Nil}(R)}$ is quasilocal with maximal ideal $\mathrm{Nil}(\phi(R))$, and $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\mathrm{Nil}(\phi(R))$.

Recall that an ideal I of a ring R is called a *divisorial ideal* of R if $(I^{-1})^{-1} = I$, where $I^{-1} = \{x \in T(R) \mid xI \subseteq R\}$. If a ring R satisfies the ascending chain condition (a.c.c.) on divisorial regular ideals of R, then R is called a *Mori ring* in the sense of [46]. An integral domain R is called a *Dedekind domain* if every nonzero ideal of R is invertible, i.e., if I is a nonzero ideal of R, then $II^{-1} = R$. If every finitely generated nonzero ideal I of an integral domain R is invertible, then R is said to be a *Prüfer domain*. If every finitely generated regular ideal of a ring R is invertible, then R is said to be a *Prüfer ring*. If R is an integral domain and $x^{-1} \in R$ for each $x \in T(R) \setminus R$, then R is called a *valuation domain*. Also, recall from [29] that an integral domain R is called a *Krull domain* if $R = \cap V_i$, where each V_i is a discrete valuation overring of R, and every nonzero element of R is a unit in all but finitely many V_i. Many characterizations and properties of Dedekind and Krull domains are given in [29], [30], and [40]. Recall from [32] that an integral domain R with quotient field K is called a *pseudo-valuation domain (PVD)* in case each prime ideal of R is *strongly prime* in the sense that $xy \in P$, $x \in K$, $y \in K$ implies that either $x \in P$ or $y \in P$. Every valuation domain is a pseudo-valuation domain. In [13], Anderson, Dobbs and the author generalized the concept of pseudo-valuation rings to the context of arbitrary rings. Recall from [13] that a prime ideal P of R is said to be *strongly prime* if either $aP \subset bR$ or $bR \subset aP$ for all $a, b \in R$. A ring R is said to be a *pseudo-valuation ring (PVR)* if every prime ideal of R is a strongly prime ideal of R.

Throughout the paper, we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the idealization of B over R is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for the "idealized ring" is $R(+)B$. See [38] for basic properties of these rings.

2 ϕ-pseudo-valuation rings and ϕ-chained rings

In [6], the author generalized the concept of pseudo-valuation domains to the context of rings that are in $\mathcal{H}$. Recall from [6] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-pseudo-valuation ring (ϕ-PVR)* if every nonnil prime ideal of R is a *ϕ-strongly prime ideal* of $\phi(R)$, in the sense that $xy \in \phi(P)$, $x \in R_{\mathrm{Nil}(R)}$, $y \in R_{\mathrm{Nil}(R)}$ (observe that $R_{\mathrm{Nil}(R)} = T(\phi(R))$) implies that either $x \in \phi(P)$ or $y \in \phi(P)$. We state some of the main results on ϕ-pseudo-valuation rings.

Theorem 2.1 ([8, Proposition 2.1]). *Let D be a PVD and suppose that P, Q are prime ideal of D such that P is properly contained in Q. Let $d \geq 1$ and choose $x \in D$ such that* $\mathrm{Rad}(xD) = P$. *Then $J = x^{d+1}D_Q$ is an ideal of D and hence D/J is a PVR* *with the following properties:*

(i) $\mathrm{Nil}(R) = P/J$ *and* $x^d \notin J$;

(ii) $Z(R) = Q/J$.

Theorem 2.2 ([8, Corollary 2.7]). *Let* $d \geq 2$, D, P, Q, x, J, *and* R *be as in Theorem 2.1. Set* $B = R_{\mathrm{Nil}(R)}$. *Then the idealization ring* $R(+)B$ *is a* ϕ*-PVR that is not a PVR.*

Theorem 2.3 ([10, Proposition 2.9], also see [23, Theorem 3.1]). *Let* $R \in \mathcal{H}$. *Then* R *is a* ϕ*-PVR if and only if* $R/\,\mathrm{Nil}(R)$ *is a PVD.*

Recall from [9] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-chained ring (ϕ-CR)* if for each $x \in R_{\mathrm{Nil}(R)} \setminus \phi(R)$ we have $x^{-1} \in \phi(R)$. A ring A is said to be a *chained ring* if for every $a, b \in A$, either $a \mid b$ (in A) or $b \mid a$ (in A).

Theorem 2.4 ([9, Corollary 2.7]). *Let* $d \geq 2$, D *be a valuation domain,* P, Q, x, J, R *be as in Theorem 2.1. Then* $R = D/J$ *is a chained ring. Furthermore, if* $B = R_{\mathrm{Nil}(R)}$, *then the idealization ring* $R(+)B$ *is a* ϕ*-CR that is not a chained ring.*

Theorem 2.5 ([9, Proposition 3.3]). *Let* $R \in \mathcal{H}$ *be a quasi-local ring with maximal ideal* M *such that* M *contains a regular element of* R . *Then* R *is a* ϕ*-PVR if and only if* $(M : M) = \{x \in T(R) \mid xM \subset M\}$ *is a* ϕ*-CR with maximal ideal* M .

Theorem 2.6 ([3, Theorem 2.7]). *Let* $R \in \mathcal{H}$. *Then* R *is a* ϕ*-CR if and only if* $R/\,\mathrm{Nil}(R)$ *is a valuation domain.*

Recall that B is said to be an *overring* of a ring A if B is a ring between A and $T(A)$.

Theorem 2.7 ([10, Corrollary 3.17]). *Let* $R \in \mathcal{H}$ *be a* ϕ*-PVR with maximal ideal* M. *The following statements are equivalent:*

Note 3 replace ; by . (at the end of items)?

(i) *Every overring of* R *is a* ϕ*-PVR;*

(ii) $R[u]$ *is a* ϕ*-PVR for each* $u \in (M : M) \setminus R$;

(iii) $R[u]$ *is quasi-local for each* $u \in (M : M) \setminus R$;

(iv) *If* B *is an overring of* R *and* $B \subset (M : M)$, *then* B *is a* ϕ*-PVR with maximal ideal* M;

(v) *If* B *is an overring of* R *and* $B \subset (M : M)$, *then* B *is quasi-local;*

(vi) *Every overring of* R *is quasi-local;*

(vii) *Every* ϕ*-CR between* R *and* $T(R)$ *other than* $(M : M)$ *is of the form* R_P *for some non-maximal prime ideal* P *of* R;

(viii) $R' = (M : M)$ *(where* R' *is the integral closure of* R *inside* $T(R)$*).*

3 Nonnil Noetherian rings (ϕ-Noetherian rings)

Recall that an ideal I of a ring R is said to be a nonnil ideal if $I \not\subseteq \mathrm{Nil}(R)$. Let $R \in \mathcal{H}$. Recall from [11] that R is said to be a a *nonnil-Noetherian ring* or just a *ϕ-Noetherian ring* as in [16] if each nonnil ideal of R is finitely generated. We have the following results.

Theorem 3.1 ([11, Corollary 2.3]). *Let $R \in \mathcal{H}$. If every nonnil prime ideal of R is finitely generated, then R is a ϕ-Noetherian ring.*

Theorem 3.2 ([11, Theorem 2.4]). *Let $R \in \mathcal{H}$. The following statements are equivalent:*

(i) *R is a ϕ-Noetherian ring;*

(ii) *$R/\mathrm{Nil}(R)$ is a Noetherian domain;*

(iii) *$\phi(R)/\mathrm{Nil}(\phi(R))$ is a Noetherian domain;*

(iv) *$\phi(R)$ is a ϕ-Noetherian ring.*

Theorem 3.3 ([11, Theorem 2.6]). *Let $R \in \mathcal{H}$. Suppose that each nonnil prime ideal of R has a power that is finitely generated. Then R is a ϕ-Noetherian ring.*

Theorem 3.4 ([11, Theorem 2.7]). *Let $R \in \mathcal{H}$. Suppose that R is a ϕ-Noetherian ring. Then any localization of R is a ϕ-Noetherian ring, and any localization of $\phi(R)$ is a ϕ-Noetherian ring.*

Theorem 3.5 ([11, Theorem 2.9]). *Let $R \in \mathcal{H}$. Suppose that R satisfies the ascending chain condition on the nonnil finitely generated ideals. Then R is a ϕ-Noetherian ring.*

Theorem 3.6 ([11, Theorem 3.4]). *Let R be a Noetherian domain with quotient field K such that $\dim(R) = 1$ and R has infinitely many maximal ideals. Then $D = R(+)K \in \mathcal{H}$ is a ϕ-Noetherian ring with Krull dimension one which is not a Noetherian ring. In particular, $\mathbb{Z}(+)\mathbb{Q}$ is a ϕ-Noetherian ring with Krull dimension one which is not a Noetherian ring (where $\mathbb{Z}$ is the set of all integer numbers with quotient field $\mathbb{Q}$).*

Theorem 3.7 ([11, Theorem 3.5]). *Let R be a Noetherian domain with quotient field K and Krull dimension $n \geq 2$. Then $D = R(+)K \in \mathcal{H}$ is a ϕ-Noetherian ring with Krull dimension n which is not a Noetherian ring. In particular, if K is the quotient field of $R = \mathbb{Z}[x_1, \ldots, x_{n-1}]$, then $R(+)K$ is a ϕ-Noetherian ring with Krull dimension n which is not a Noetherian ring.*

In the following result, we show that a ϕ-Noetherian ring is related to a pullback of a Noetherian domain.

Theorem 3.8 ([16, Theorem 2.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Noetherian ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & S = A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring containing A with maximal ideal M, $S = A/M$ is a Noetherian subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Theorem 3.9 ([16, Proposition 2.4]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let $I \neq R$ be an ideal of R. If $I \subset \mathrm{Nil}(R)$, then R/I is a ϕ-Noetherian ring. If $I \not\subset \mathrm{Nil}(R)$, then $\mathrm{Nil}(R) \subset I$ and R/I is a Noetherian ring. Moreover, if $\mathrm{Nil}(R) \subset I$, then R/I is both Noetherian and ϕ-Noetherian if and only if I is either a prime ideal or a primary ideal whose radical is a maximal ideal.*

Theorem 3.10 ([16, Corollary 2.5]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then a homomorphic image of R is either a ϕ-Noetherian ring or a Noetherian ring.*

Our next result shows that a ϕ-Noetherian ring satisfies the conclusion of the Principal Ideal Theorem (and the Generalized Principal Ideal Theorem).

Theorem 3.11 ([16, Theorem 2.7]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime ideal. If P is minimal over an ideal generated by n or fewer elements, then the height of P is less than or equal to n. In particular, each prime minimal over a nonnil element of R has height one.*

Other statements about primes of Noetherian rings that can be easily adapted to statements about primes of ϕ-Noetherian rings include the following.

Note 4 replaced satisfies by satisfy

Theorem 3.12 ([16, Proposition 2.8] and [40, Theorem 145]). *Let $R \in \mathcal{H}$ satisfy the ascending chain condition on radical ideals. If R has an infinite number of prime ideals of height one, then their intersection is* $\mathrm{Nil}(R)$.

Theorem 3.13 ([16, Proposition 2.9]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and P be a nonnil prime ideal of R of height n. Then there exist nonnil elements $a_1, \ldots, a_n$ in R such that P is minimal over the ideal $(a_1, \ldots, a_n)$ of R, and for any i $(1 \le i \le n)$, every (nonnil) prime ideal of R minimal over $(a_1, \ldots, a_i)$ has height i.*

Theorem 3.14 ([16, Proposition 2.10]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let I be an ideal of R generated by n elements with $I \neq R$. If P is a prime ideal containing I with P/I of height k, then the height of P is less than or equal to $n + k$.*

Theorem 3.15 ([16, Proposition 3.1]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x]$ that contracts to P but properly contains $PR[x]$, then $PR[x]$ has height n and Q has height $n + 1$.*

Similar height restrictions exist for the primes of $R[x_1, \ldots, x_m]$.

Theorem 3.16 ([16, Proposition 3.2]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a height n prime of R. If Q is a prime of $R[x_1, \ldots, x_m]$ that contracts to P but properly contains $PR[x_1, \ldots, x_m]$, then $PR[x_1, \ldots, x_m]$ has height n and Q has height at most $n + m$. Moreover the prime $PR[x_1, \ldots, x_m] + (x_1, \ldots, x_m)R[x_1, \ldots, x_m]$ has height $n + m$.*

Theorem 3.17 ([16, Corollary 3.3]). *If R is a finite dimensional ϕ-Noetherian ring of dimension n, then $\dim(R[x_1, \ldots, x_m]) = n + m$ for each integer $m > 0$.*

In our next result, we show that each ideal of $R[x]$ that contracts to a nonnil ideal of R is finitely generated.

Theorem 3.18 ([16, Proposition 3.4]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. If I is an ideal of $R[x_1, \ldots, x_n]$ for which $I \cap R$ is not contained in $\mathrm{Nil}(R)$, then I is a finitely generated ideal of $R[x_1, \ldots, x_n]$.*

Since three distinct comparable primes of $R[x]$ cannot contract to the same prime of R, a consequence of Theorem 3.18 is that the search for primes of $R[x]$ that are not finitely generated can be restricted to those of height one. A similar statement can be made for primes of $R[x_1, \ldots, x_n]$.

Theorem 3.19 ([16, Corollary 3.5]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring and let P be a prime of $R[x_1, \ldots, x_n]$. If P has height greater than n, then P is finitely generated.*

The ring in our next example shows that the converse of Theorem 3.18 does not hold even for prime ideals.

Example 3.20 ([16, Example 3.6]). Let $R = D(+)L$ be the idealization of $L = K((y))/D$ over $D = K[[y]]$. Then R is a quasilocal ϕ-Noetherian ring with nilradical $\mathrm{Nil}(R)$ isomorphic to L. Consider the polynomial $g(x) = 1 - yx$. Since the coefficients of g generate D as an ideal and g is irreducible, $P = gD[x]$ is a height-one principal prime of $D[x]$ with $P \cap D = (0)$. Each nonzero element of L can be written in the form d/y^n where n is a positive integer, y denotes the image of y in L and $d = d_0 + d_1 y + \cdots + d_{n-1}y^{n-1}$ with $d_0 \neq 0$. Given such an element, let $f(x) = 1 + yx + \cdots + y^{n-1}x^{n-1} \in L[x]$. Then $g(x)(df(x)/y^n) = d/y^n$ since $dy^n/y^n = 0$ in L. It follows that $g(x)R[x]$ is a height-one principal prime of $R[x]$ that contracts to $\mathrm{Nil}(R)$.

4 ϕ-Prüfer rings and ϕ-Bezout rings

We say that a nonnil ideal I of R is *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [3] that R is called a *ϕ-Prüfer ring* if every finitely generated nonnil ideal of R is ϕ-invertible.

Theorem 4.1 ([3, Corollary 2.10]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Prüfer ring;*

(ii) *$\phi(R)$ is a Prüfer ring;*

(iii) *$\phi(R)/\operatorname{Nil}(\phi(R))$ is a Prüfer domain;*

(iv) *R_P is a ϕ-CR for each prime ideal P of R;*

(v) *$R_P/\operatorname{Nil}(R_P)$ is a valuation domain for each prime ideal P of R;*

(vi) *$R_M/\operatorname{Nil}(R_M)$ is a valuation domain for each maximal ideal M of R;*

(vii) *R_M is a ϕ-CR for each maximal ideal M of R.*

Theorem 4.2 ([3, Theorem 2.11]). *Let $R \in \mathcal{H}$ be a ϕ-Prüfer ring and let S be a ϕ-chained overring of R. Then $S = R_P$ for some prime ideal P of R containing $Z(R)$.*

The following is an example of a ring $R \in \mathcal{H}$ such that R is a Prüfer ring, but R is not a ϕ-Prüfer ring.

Example 4.3 ([3, Example 2.15]). Let $n \geq 1$ and let D be a non-integrally closed domain with quotient field K and Krull dimension n. Set $R = D(+)(K/D)$. Then $R \in \mathcal{H}$ and R is a Prüfer ring with Krull dimension n which is not a ϕ-Prüfer ring.

Theorem 4.4 ([3, Theorem 2.17]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if every overring of $\phi(R)$ is integrally closed.*

Example 4.5 ([3, Example 2.18]). Let $n \geq 1$ and let D be a Prüfer domain with quotient field K and Krull dimension n. Set $R = D(+)K$. Then $R \in \mathcal{H}$ is a (non-domain) ϕ-Prüfer ring with Krull dimension n.

Recall from [21] that a ring R is said to be a *pre-Prüfer ring* if R/I is a Prüfer ring for every nonzero proper ideal I of R.

Theorem 4.6 ([3, Theorem 2.19]). *Let $R \in \mathcal{H}$ such that* $\operatorname{Nil}(R) \neq \{0\}$. *Then R is a pre-Prüfer ring if and only if R is a ϕ-Prüfer ring.*

The following example shows that the hypothesis $\operatorname{Nil}(R) \neq \{0\}$ in Theorem 4.6 is crucial.

Example 4.7 ([3, Example 2.20] and [42, Example 2.9]). Let D be a Prüfer domain with quotient field F. For indeterminates X, Y, let $K = F(Y)$ and let V be the valuation domain $K + XK[[X]]$. Then V is one-dimensional with maximal ideal $M = XK[[X]]$. Set $R = D + M$. Then $\operatorname{Nil}(R) = \{0\}$, and R is a pre-Prüfer ring (domain) which is not a Prüfer ring (domain). Hence R is not a ϕ-Prüfer ring.

Note 5 replaced Besout by Bezout

Recall from [3] that a ring $R \in \mathcal{H}$ is said to be a *ϕ-Bezout ring* if $\phi(I)$ is a principal ideal of $\phi(R)$ for every finitely generated nonnil ideal I of R. A ϕ-Bezout ring is a ϕ-Prüfer ring, but of course the converse is not true. A ring R is said to be a *Bezout ring* if every finitely generated regular ideal of R is principal.

Theorem 4.8 ([3, Corollary 3.5]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Bezout ring;*

(ii) *$R/\operatorname{Nil}(R)$ is a Bezout domain;*

(iii) *$\phi(R)/\operatorname{Nil}(\phi(R))$ is a Bezout domain;*

(iv) *$\phi(R)$ is a Bezout ring;*

(v) *Every finitely generated nonnil ideal of R is principal.*

Theorem 4.9 ([3, Theorem 3.9]). *Let $R \in \mathcal{H}$ be quasi-local. Then R is a ϕ-CR if and only if R is a ϕ-Bezout ring.*

Example 4.10 ([3, Example 3.8]). Let $n \geq 1$ and let D be a Bezout domain with quotient field K and Krull dimension n. Set $R = D(+)K$. Then $R \in \mathcal{H}$ is a (nondomain) ϕ-Bezout ring with Krull dimension n.

5 ϕ-Dedekind rings

Let $R \in \mathcal{H}$. We say that a nonnil ideal I of R is *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$. If every nonnil ideal of R is ϕ-invertible, then we say that R is a *ϕ-Dedekind ring*.

Theorem 5.1 ([4, Theorem 2.6]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Dedekind subring of T/M, the verical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Example 5.2 ([4, Example 2.7]). Let D be a Dedekind domain with quotient field K, and let L be an extension ring of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Dedekind ring which is not a Dedekind domain.

We say that a ring $R \in \mathcal{H}$ is *ϕ-(completely) integrally closed* if $\phi(R)$ is (completely) integrally closed in $T(\phi(R)) = R_{\operatorname{Nil}(R)}$. The following characterization of ϕ-Dedekind rings resembles that of Dedekind domains as in [40, Theorem 96].

Theorem 5.3 ([4, Theorem 2.10]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is ϕ-Dedekind;*

(ii) *R is nonnil-Noetherian (ϕ-Noetherian), ϕ-integrally closed, and of dimension ≤ 1;*

(iii) *R is nonnil-Noetherian and R_M is a discrete ϕ-chained ring for each maximal ideal M of R.*

A ring R is said to be a *Dedekind ring* if every nonzero ideal of R is invertible.

Theorem 5.4 ([4, Theorem 2.12]). *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring. Then R is a Dedekind ring.*

The following is an example of a ring $R \in \mathcal{H}$ which is a Dedekind ring but not a ϕ-Dedekind ring.

Example 5.5 ([4, Example 2.13]). Let D be a non-Dedekind domain with (proper) quotient field K. Set $R = D(+)K/D$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence R is a Dedekind ring. Since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D, R is not a ϕ-Dedekind ring by [4, Theorem 2.5].

It is well known that an integral domain R is a Dedekind domain iff every nonzero proper ideal of R is (uniquely) a product of prime ideals of R. We have the following result.

Theorem 5.6 ([4, Theorem 2.15]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Dedekind ring if and only if every nonnil proper ideal of R is (uniquely) a product of nonnil prime ideals of R.*

Theorem 5.7 ([4, Theorem 2.16]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Dedekind ring;*

(ii) *Each nonnil proper principal ideal aR can be written in the form $aR = Q_1 \cdots Q_n$, where each Q_i is a power of a nonnil prime ideal of R and the Q_i's are pairwise comaximal;*

(iii) *Each nonnil proper ideal I of R can be written in the form $I = Q_1 \cdots Q_n$, where each Q_i is a power of a nonnil prime ideal of R and the Q_i's are pairwise comaximal.*

Theorem 5.8 ([4, Theorem 2.20]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Dedekind ring;*

(ii) *Each nonnil prime ideal of R is ϕ-invertible;*

(iii) *R is a nonnil-Noetherian ring and each nonnil maximal ideal of R is ϕ-invertible.*

Theorem 5.9 ([4, Theorem 2.23]). *Let $R \in \mathcal{H}$ be a ϕ-Dedekind ring. Then every overring of R is a ϕ-Dedekind ring.*

6 Factoring nonnil ideals into prime and invertible ideals

In this section, we give a generalization of the concept of factorization of ideals of an integral domain into a finite product of invertible and prime ideals which was extensively studied by Olberding [48] to the context of rings that are in the class $\mathcal{H}$. Observe that if R is an integral domain, then $R \in \mathcal{H}$. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq \mathrm{Nil}(R)$. Let $R \in \mathcal{H}$. Then R is said to be a *ϕ-ZPUI ring* if each nonnil ideal I of $\phi(R)$ can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, \ldots, P_n$ are prime ideals of $\phi(R)$. If every nonnil ideal I of R can be written as $I = JP_1 \cdots P_n$, where J is an invertible ideal of R and $P_1, \ldots, P_n$ are prime ideals of R, then R is said to be a *nonnil-ZPUI ring*. Commutative ϕ-ZPUI rings that are in $\mathcal{H}$ are characterized in [12, Theorem 2.9]. Examples of ϕ-ZPUI rings that are not ZPUI rings are constructed in [12, Theorem 2.13]. It is shown in [12, Theorem 2.14] that a ϕ-ZPUI ring is the pullback of a ZPUI domain. It is shown in [12, Theorem 3.1] that a nonnil-ZPUI ring is a ϕ-ZPUI ring. Examples of ϕ-ZPUI rings that are not nonnil-ZPUI rings are constructed in [12, Theorem 3.2]. We call a ring $R \in \mathcal{H}$ a *nonnil-strongly discrete ring* if R has no nonnil prime ideal P such that $P^2 = P$. A ring $R \in \mathcal{H}$ is said to be *nonnil-h-local* if each nonnil ideal of R is contained in at most finitely many maximal ideals of R and each nonnil prime ideal P of R is contained in a unique maximal ideal of R.

Since the class of integral domains is a subset of $\mathcal{H}$, the following result is a generalization of [48, Theorem 2.3].

Theorem 6.1 ([12, Theorem 2.9]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-ZPUI ring;*

(ii) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R;*

(iii) *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$;*

(iv) *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*

In the following result, we show that a nonnil-ZPUI ring is a ϕ-ZPUI ring.

Theorem 6.2 ([12, Theorem 3.1]). *Let $R \in \mathcal{H}$ be a nonnil-ZPUI ring. Then R is a ϕ-ZPUI ring, and hence all the following statements hold:*

Note 6 replaced period by colon

(i) *$R/\mathrm{Nil}(R)$ is a ZPUI domain.*

(ii) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.*

(iii) *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.*

(iv) *R is a nonnil-strongly discrete nonnil-h-local ϕ-Prüfer ring.*

(v) *R is a nonnil-strongly discrete nonnil-h-local Prüfer ring.*

Examples of ϕ-ZPUI rings that are not nonnil-ZPUI rings are constructed in the following result.

Theorem 6.3 ([12, Theorem 3.2]). *Let A be a ZPUI domain that is not a Dedekind domain with Krull dimension $n \geq 1$ and quotient field K. Then $R = A(+)K/A \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n which is not a nonnil-ZPUI ring.*

Olberding in [48, Corollary 2.4] showed that for each $n \geq 1$, there exists a ZPUI domain with Krull dimension n. A Dedekind domain is a trivial example of a ZPUI domain. We have the following result.

Theorem 6.4 ([12, Theorem 2.13]). *Let A be a ZPUI domain (i.e. A is a strongly discrete h-local Prüfer domain by* [48, Theorem2.3]*) with Krull dimension $n \geq 1$ and quotient field F, and let K be an extension ring of F (i.e. K is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n that is not a ZPUI ring.*

In the following result, we show that a ϕ-ZPUI ring is the pullback of a ZPUI domain. A good paper for pullbacks is the article by Fontana [27].

Theorem 6.5 ([12, Theorem 2.14]). *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a ZPUI ring that is a subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

7 ϕ-Krull rings

We say that a ring $R \in \mathcal{H}$ is a *discrete ϕ-chained ring* if R is a ϕ-chained ring with at most one nonnil prime ideal and every nonnil ideal of R is principal. Recall from [4] that a ring $R \in \mathcal{H}$ is said to be a ϕ-Krull ring if $\phi(R) = \cap V_i$, where each V_i is a discrete ϕ-chained overring of $\phi(R)$, and for every nonnilpotent element $x \in R$, $\phi(x)$ is a unit in all but finitely many V_i.

Theorem 7.1 ([4, Theorem 3.1]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $R/\operatorname{Nil}(R)$ is a Krull domain.*

We have the following pullback characterization of ϕ-Krull rings.

Theorem 7.2 ([4, Theorem 3.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a Krull subring of T/M, the verical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Example 7.3 ([4, Example 3.3]). Let D be a Krull domain with quotient field K, and let L be a ring extension of K. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-Krull ring which is not a Krull domain.

It is well known [29, Theorem 3.6] that an integral domain R is a Krull domain if and only if R is a completely integrally closed Mori domain. We have a similar characterization for ϕ-Krull rings.

Theorem 7.4 ([4, Theorem 3.4]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Krull ring if and only if R is a ϕ-completely integrally closed ϕ-Mori ring.*

Theorem 7.5 ([4, Theorem 3.5]). *Let $R \in \mathcal{H}$ be a ϕ-Krull ring which is not zero-dimensional. Then the following statements are equivalent:*

(i) *R is a ϕ-Prüfer ring;*

(ii) *R is a ϕ-Dedekind ring;*

(iii) *R is one-dimensional.*

It is well known that if R is a Noetherian domain, then R' is a Krull domain. In particular, an integrally closed Noetherian domain is a Krull domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 7.6 ([4, Theorem 3.6]). *Let $R \in \mathcal{H}$ be a nonnil-Noetherian ring. Then $\phi(R)'$ is a ϕ-Krull ring. In particular, if R is a ϕ-integrally closed nonnil-Noetherian ring, then R is a ϕ-Krull ring.*

It is known [40, Problem 8, page 83] that if R is a Krull domain in which all prime ideals of height ≥ 2 are finitely generated, then R is a Noetherian domain. We have the following analogous result for nonnil-Noetherian rings.

Theorem 7.7 ([4, Theorem 3.7]). *Let $R \in \mathcal{H}$ be a ϕ-Krull ring in which all prime ideals of R with height ≥ 2 are finitely generated. Then R is a nonnil-Noetherian ring.*

For a ring $R \in \mathcal{H}$, let ϕ_R denotes the ring-homomorphism $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$. It is well known [29, Proposition 1.9, page 8] that an integral domain R is a Krull domain if and only if R satisfies the following three conditions:

(i) R_P is a discrete valuation domain for every height-one prime ideal P of R;

(ii) $R = \cap R_P$, the intersection being taken over all height-one prime ideals P of R;

(iii) Each nonzero element of R is in only a finite number of height-one prime ideals of R, i.e., each nonzero element of R is a unit in all but finitely many R_P, where P is a height-one prime ideal of R.

The following result is an analog of [29, Proposition 1.9, page 8].

Theorem 7.8 ([4, Theorem 3.9]). *Let $R \in \mathcal{H}$ with* $\dim(R) \geq 1$. *Then R is a ϕ-Krull ring if and only if R satisfies the following three conditions:*

(i) *R_P is a discrete ϕ-chained ring for every height-one prime ideal P of R;*

(ii) *$\phi_R(R) = \cap \phi_{R_P}(R_P)$, the intersection being taken over all height-one prime ideals P of R;*

(iii) *Each nonnilpotent element of R lies in only a finite number of height-one prime ideals of R, i.e., each nonnilpotent element of R is a unit in all but finitely many R_P, where P is a height-one prime ideal of R.*

Recall that a ring R is called a *Marot ring* if each regular ideal of R is generated by its set of regular elements. A Marot ring is called a *Krull ring* in the sense of [38, page 37] if either $R = T(R)$ or if there exists a family $\{V_i\}$ of discrete rank-one valuation rings such that:

(i) R is the intersection of the valuation rings $\{V_i\}$;

(ii) Each regular element of $T(R)$ is a unit in all but finitely many V_i.

The following is an example of a ring $R \in \mathcal{H}$ which is a Krull ring but not a ϕ-Krull ring.

Example 7.9 ([4, Example 3.12]). Let D be a non-Krull domain with (proper) quotient field K. Set $R = D(+)K/D$. Then $R \in \mathcal{H}$ and $R = T(R)$. Hence R is a Krull ring. Since $R/\mathrm{Nil}(R)$ is ring-isomorphic to D, R is not a ϕ-Krull ring by Theorem 7.1.

8 ϕ-Mori rings

According to [46], a ring R is called a *Mori ring* if it satisfies a.c.c. on divisorial regular ideals. Let $R \in \mathcal{H}$. A nonnil ideal I of R is ϕ-divisorial if $\phi(I)$ is a divisorial ideal of $\phi(R)$, and R is a *ϕ-Mori ring* if it satisfies a.c.c. on ϕ-divisorial ideals.

The following is a characterization of ϕ-Mori rings in terms of Mori rings in the sense of [46].

Theorem 8.1 ([17, Theorem 2.2]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $\phi(R)$ is a Mori ring.*

The following is a characterization of ϕ-Mori rings in terms of Mori domains.

Theorem 8.2 ([17, Theorem 2.5]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $R/\operatorname{Nil}(R)$ is a Mori domain.*

Theorem 8.3 ([17, Theorem 2.7]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring. Then R satisfies a.c.c. on nonnil divisorial ideals of R. In particular, R is a Mori ring.*

The converse of Theorem 8.3 is not valid as it can be seen by the following example.

Example 8.4 ([17, Example 2.8]). Let D be an integral domain with quotient field L which is not a Mori domain and set $R = D(+)(L/D)$, the idealization of L/D over D. Then $R \in \mathcal{H}$ is a Mori ring which is not a ϕ-Mori ring.

Example 8.18 shows how to construct a nontrivial Mori ring (i.e., where $R \neq T(R)$) in $\mathcal{H}$ which is not ϕ-Mori.

Theorem 8.5 ([17, Theorem 2.10]). *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then R is both a ϕ-Mori ring and a Mori ring.*

Given a Krull domain of the form $E = L + M$, where L is a field and M a maximal ideal of E, any subfield K of L gives rise to a Mori domain $D = K + M$. If L is not a finite algebraic extension of K, then D cannot be Noetherian (see [19, Section 4]). We make use of this in our next example to build a ϕ-Mori ring which is neither an integral domain nor a ϕ-Noetherian.

Example 8.6 ([17, Example 2.11]). Let K be the quotient field of the ring $D = \mathbb{Q} + X\mathbb{R}[[X]]$ and set $R = D(+)K$, the idealization of K over D. It is easy to see that $\operatorname{Nil}(R) = \{0\}(+)K$ is a divided prime ideal of R. Hence $R \in \mathcal{H}$. Now since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D and D is a Mori domain but not a Noetherian domain, we conclude that R is a ϕ-Mori ring which is not a ϕ-Noetherian ring.

In light of Example 8.6, ϕ-Mori rings can be constructed as in the following example.

Example 8.7 ([17, Example 2.12]). Let D be a Mori domain with quotient field K and let L be an extension ring of K. Then $R = D(+)L$, the idealization of L over D, is in $\mathcal{H}$. Moreover, R is a ϕ-Mori ring since $R/\operatorname{Nil}(R)$ is ring-isomorphic to D which is a Mori domain.

The following result is a generalization of [54, Theorem 1]. An analogous result holds for Mori rings when the chains under consideration are restricted to regular divisorial ideals whose intersection is regular [46, Theorem 2.22].

Theorem 8.8 ([17, Theorem 2.13]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if whenever $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq \mathrm{Nil}(R)$, then $\{I_m\}$ is a finite set.*

Let D be an integral domain with quotient field K. If I is an ideal of D, then $(D : I) = \{x \in K \mid xI \subseteq D\}$. Mori domains can be characterized by the property that for each nonzero ideal I, there is a finitely generated ideal $J \subset I$ such that $(D : I) = (D : J)$ (equivalently, $I_v = J_v$) ([51, Theorem 1]). Our next result generalizes this result to ϕ-Mori rings.

Theorem 8.9 ([17, Theorem 2.14]). *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if for any nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$, equivalently, $\phi(J)_v = \phi(I)_v$.*

In the following theorem we combine all of the different characterizations of ϕ-Mori rings stated in this section.

Theorem 8.10 ([17, Corollary 2.15]). *Let $R \in \mathcal{H}$. The following statements are equivalent:*

(i) *R is a ϕ-Mori ring;*

(ii) *$R/\mathrm{Nil}(R)$ is a Mori domain;*

(iii) *$\phi(R)/\mathrm{Nil}(\phi(R))$ is a Mori domain;*

(iv) *$\phi(R)$ is a Mori ring.*

(v) *If $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq \mathrm{Nil}(R)$, then $\{I_m\}$ is a finite set;*

(vi) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$;*

(vii) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)_v = \phi(I)_v$.*

The following result is a generalization of [54, Theorem 5].

Theorem 8.11 ([17, Theorem 3.1]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and I be a nonzero ϕ-divisorial ideal of R. Then I contains a power of its radical.*

We recall a few definitions regarding special types of ideals in integral domains. For a nonzero ideal I of an integral domain D, I is said to be strong if $II^{-1} = I$, strongly divisorial if it is both strong and divisorial, and v-invertible if $(II^{-1})_v = D$. We will extend these concepts to the rings in $\mathcal{H}$.

Let I be a nonnil ideal of a ring $R \in \mathcal{H}$. We say that I is *strong* if $II^{-1} = I$, *ϕ-strong* if $\phi(I)\phi(I)^{-1} = \phi(I)$, *strongly divisorial* if it is both strong and divisorial, *strongly ϕ-divisorial* if it is both ϕ-strong and ϕ-divisorial, *v-invertible* if $(II^{-1})_v = R$

and *ϕ-v-invertible* if $(\phi(I)\phi(I)^{-1})_v = \phi(R)$. Obviously, I is ϕ-strong, strongly ϕ-divisorial or ϕ-v-invertible if and only if $\phi(I)$ is, respectively, strong, strongly divisorial or v-invertible.

In [51, Proposition 1], J. Querré proved that if P is a prime ideal of a Mori domain D, then P is divisorial when it is height one. In the same proposition, he incorrectly asserted that if the height of P is larger than one and P^{-1} properly contains D, then P is strongly divisorial. While it is true that such a prime must be strong, a (Noetherian) counterexample to the full statement can be found in [34]. What one can say is that P_v will be strongly divisorial (see [5]).

Theorem 8.12 ([17, Theorem 3.3]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and P be a (nonnil) prime ideal of R. If* $\mathrm{ht}(P) = 1$*, then P is ϕ-divisorial. If* $\mathrm{ht}(P) \geq 2$*, then either $\phi(P)^{-1} = \phi(R)$ or $\phi(P)_v$ is strongly divisorial.* **Note 7** ht upright

For a ϕ-Mori ring $R \in \mathcal{H}$, let $D_m(R)$ denote the *maximal ϕ-divisorial ideals of R*; i.e., the set of nonnil ideals of R maximal with respect to being ϕ-divisorial. The following result generalizes [25, Theorem 2.3] and [19, Proposition 2.1].

Theorem 8.13 ([17, Theorem 3.4]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring such that* $\mathrm{Nil}(R)$ *is not the maximal ideal of R. Then the following hold:* **Note 8** replaced period by colon

(a) *The set $\mathcal{D}_m(R)$ is nonempty. Moreover, $M \in \mathcal{D}_m(R)$ if and only if $M/\mathrm{Nil}(R)$ is a maximal divisorial ideal of $R/\mathrm{Nil}(R)$.*

(b) *Every ideal of $\mathcal{D}_m(R)$ is prime.*

(c) *Every nonnilpotent nonunit element of R is contained in a finite number of maximal ϕ-divisorial ideals.*

As with a nonempty subset of R, a nonempty set of ideals $\mathbb{S}$ is *multiplicative* if (i) the zero ideal is not contained in $\mathbb{S}$, and (ii) for each I and J in $\mathbb{S}$, the product IJ is in $\mathcal{S}$. Such a set $\mathbb{S}$ is referred to as a multiplicative system of ideals and it gives rise to a generalized ring of quotients $R_{\mathbb{S}} = \{t \in T(R) \mid tI \subset R$ for some $I \in \mathcal{S}\}$. For each prime ideal P, $R_{(P)} = \{t \in T(R) \mid st \in R$ for some $s \in R\backslash P\} = R_{\mathbb{S}}$, where $\mathbb{S}$ is the set of ideals (including R) that are not contained in P. Note that in general a localization of a Mori ring need not be Mori (see Example 8.18 below). On the other hand, if $\mathbb{S}$ is a multiplicative system of regular ideals, then $R_{\mathbb{S}}$ is a Mori ring whenever R is Mori ring ([46, Theorem 2.13]).

Theorem 8.14 ([17, Theorem 3.5], and [17, Theorem 2.2]). *Let R be a ϕ-Mori ring. Then*

(a) *R_S is a ϕ-Mori ring for each multiplicative set S.*

(b) *R_P is a ϕ-Mori ring for each prime P.*

(c) *$R_{\mathcal{S}}$ is a ϕ-Mori ring for each multiplicative system of ideals $\mathcal{S}$.*

(d) *$R_{(P)}$ is a ϕ-Mori ring for each prime ideal P.*

One of the well-known characterizations of Mori domains is that an integral domain D is a Mori domain if and only if (i) D_M is a Mori domain for each maximal divisorial ideal M, (ii) $D = \cap D_M$ where the M range over the set of maximal divisorial ideals of D, and (iii) each nonzero element is contained in at most finitely many maximal divisorial ideals ([52, Théorème 2.1] and [54, Théorème I.2]). A similar statement holds for ϕ-Mori rings. Note that in condition (ii), if D has no maximal divisorial ideals, the intersection is assumed to be the quotient field of D. For the equivalence, that means that D is its own quotient field. The analogous statement is that if $\mathcal{D}_m$ is empty, then we have $R = T(R) = R_{\mathrm{Nil}(R)}$ with $\mathrm{Nil}(R)$ the maximal ideal.

Note 9 added 'statements'

Theorem 8.15 ([17, Theorem 3.6]). *Let $R \in \mathcal{H}$. Then the following statements are equivalent:*

(i) *R is a ϕ-Mori ring;*

Note 10 replaced (i) etc. by (a) etc.; add some wording to improve line break?

(ii) (a) *R_M is a ϕ-Mori ring for each maximal ϕ-divisorial M,* (b) *$\phi(R) = \cap\, \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and* (c) *each non-nil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals;*

(iii) (a) *$R_{(M)}$ is a ϕ-Mori ring for each maximal ϕ-divisorial M,* (b) *$\phi(R) = \cap\, \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and* (c) *each non-nil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals.*

In [19], V. Barucci and S. Gabelli proved that if P is a maximal divisorial ideal of a Mori domain D, then the following three conditions are equivalent: (1) D_P is a discrete rank-one valuation domain, (2) P is v-invertible, and (3) P is not strong [19, Theorem 2.5]. A similar result holds for ϕ-Mori rings.

Theorem 8.16 ([17, Theorem 3.9]). *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and $P \in \mathcal{D}_m(R)$. Then the following statements are equivalent:*

(i) *R_P is a discrete rank-one ϕ-chained ring;*

(ii) *P is ϕ-v-invertible;*

(iii) *P is not ϕ-strong.*

Recall from [38] that if $f(x) \in R[x]$, then $c(f)$ denotes the ideal of R generated by the coefficients of $f(x)$, and $R(x)$ denotes the quotient ring $R[x]_S$ of the polynomial ring $R[x]$, where S is the set of $f \in R[x]$ such that $c(f) = R$.

Note 11 replace for which by with to improve line break?; added 'statements'; may we replace (1) etc. by (i) etc.?

Theorem 8.17 ([17, Theorem 4.5]). *Let R be an integrally closed ring for which $\mathrm{Nil}(R) = Z(R) \neq \{0\}$. Then the following statements are equivalent:*

(1) *R is ϕ-Mori and the nilradical of $T(R[x])$ is an ideal of $R(x)$;*

(2) *$R(x)$ is ϕ-Mori;*

(3) *$R(x)$ is ϕ-Noetherian;*

(4) *R is ϕ-Noetherian and the nilradical of $T(R[x])$ is an ideal of $R(x)$;*

(5) *Each regular ideal of R is invertible;*

(6) *$R/\operatorname{Nil}(R)$ is a Dedekind domain;*

(7) *R is a ϕ-Dedekind ring.*

As mentioned above, a Mori ring is said to be nontrivial if it is properly contained in its total quotient ring. Our next example is of a nontrival Mori ring that is in the set $\mathcal{H}$ but is not a ϕ-Mori ring.

Example 8.18 ([17, Example 5.3]). Let E be a Dedekind domain with a maximal ideal M such that no power of M is principal (equivalently, M generates an infinite cyclic subgroup of the class group) and let $D = E + xF[x]$, where F is the quotient field of E. Let $\mathcal{P} = \{ND \mid N \in \operatorname{Max}(E)\backslash\{M\}\}$, $B = \sum F/D_{P_\alpha}$ where each $P_\alpha \in \mathcal{P}$, and let $R = D(+)B$. Then the following hold: **Note 12** replaced period by colon

(a) If J is a regular ideal, then $J = I(+)B = IR$ for some ideal I that contains a polynomial in D whose constant term is a unit of E. Moreover, the ideal I is principal and factors uniquely as $P_1^{r_1} \cdots P_n^{r_n}$, where the P_i are the height-one maximal ideals of D that contain I.

(b) $R \neq T(R)$ since, for example, the element $(1 + x, 0)$ is a regular element of R that is not a unit.

(c) R is a nontrivial Mori ring but R is not ϕ-Mori.

(d) MR is a maximal ϕ-divisorial ideal of R, but R_{MR} is not a Mori ring.

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